$$V = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}(\sec x - \tan x)^2\, dx$$
$$= \frac{\pi}{4}\int_{-\pi/3}^{\pi/3}(\sec^2 x - 2\sec x \tan x + \tan^2 x)\, dx$$
$$= \frac{\pi}{4}\left[\tan x - 2\sec x + \tan x - x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\tan x - \sec x - \frac{1}{2}x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\left(\sqrt{3} - 2 - \frac{\pi}{6}\right) - \left(-\sqrt{3} - 2 + \frac{\pi}{6}\right)\right]$$
$$= \pi\sqrt{3} - \frac{\pi^2}{6}.$$

(b) $A(x) = s^2 = w^2 = (\sec x - \tan x)^2$, and $V = \int_{-\pi/3}^{\pi/3}(\sec x - \tan x)^2\, dx$, which by same method as in part (a) equals $4\sqrt{3} - \frac{2}{3}\pi.$

41. A cross section has width $w = \sqrt{5}y^2$ and area $\pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \frac{5\pi}{4}y^4$. The volume is $\int_0^2 \frac{5\pi}{4}y^4 dy = \frac{\pi}{4}\left[y^5\right]_0^2 = 8\pi.$

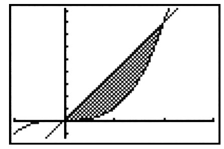
42. A cross section has width $w = 2\sqrt{1-y^2}$ and area $\frac{1}{2}s^2 = \frac{1}{2}w^2 = 2(1-y^2)$. This volume is $\int_{-1}^{1} 2(1-y^2)\, dy = 2\left[y - \frac{1}{3}y^3\right]_{-1}^{1} = \frac{8}{3}.$

43. Since the diameter of the circular base of the solid extends from $y = \frac{12}{2} = 6$ to $y = 12$, for a diameter of 6 and a radius of 3, the solid has the same cross sections as the right circular cone. The volumes are equal by Cavalieri's Theorem.

44. **(a)** The volume is the same as if the square had moved without twisting: $V = Ah = s^2h.$

(b) Still s^2h: the lateral distribution of the square cross sections doesn't affect the volume. That's Cavalieri's Volume Theorem.

45.



[–1, 3] by [–1.4, 9.1]

The functions intersect at (2, 8).

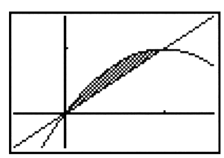
(a) Use washer cross sections: a washer has inner radius $r = x^3$, outer radius $R = 4x$, and area $A(x) = \pi(R^2 - r^2) = \pi(16x^2 - x^6)$. The volume is

$$\int_0^2 \pi(16x^2 - x^6)\, dx = \pi\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2 = \frac{512\pi}{21}.$$

(b) Use cylindrical shells: a shell has a radius $8 - y$ and height $y^{1/3} - \frac{y}{4}$. The volume is

$$\int_0^8 2\pi(8-y)\left(y^{1/3} - \frac{y}{4}\right)dy$$
$$= 2\pi\int_0^8\left(8y^{1/3} - 2y - y^{4/3} + \frac{y^2}{4}\right)dy$$
$$= 2\pi\left[6y^{4/3} - y^2 - \frac{3}{7}y^{7/3} + \frac{1}{12}y^3\right]_0^8$$
$$= \frac{832\pi}{21}.$$

46.



[–0.5, 1.5] by [–0.5, 1.5]

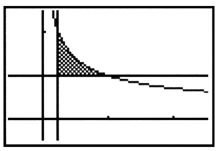
The functions intersect at (0, 0) and (1, 1).

(a) Use cylindrical shells: A shell has radius x and height $2x - x^2 - x = x - x^2$. The volume is

$$\int_0^1 2\pi(x)(x - x^2)\, dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6}.$$

(b) Use cylindrical shells: a shell has radius $1-x$ and height $2x-x^2-x=x-x^2$. The volume is

$$\int_0^1 2\pi(1-x)(x-x^2)dx$$
$$=2\pi\int_0^1(x^3-2x^2+x)\,dx$$
$$=2\pi\left[\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^1$$
$$=\frac{\pi}{6}.$$

47. 

[−0.5, 2.5] by [−0.5, 2.5]

The intersection points are $\left(\frac{1}{4}, 1\right), \left(\frac{1}{4}, 2\right)$, and (1, 1).

(a) A washer has inner radius $r=\frac{1}{4}$, outer radius $R=\frac{1}{y^2}$, and area $\pi(R^2-r^2)=\pi\left(\frac{1}{y^4}-\frac{1}{16}\right)$. The volume is

$$\int_1^2 \pi\left(\frac{1}{y^4}-\frac{1}{16}\right)dy=\pi\left[-\frac{1}{3y^3}-\frac{1}{16}y\right]_1^2$$
$$=\frac{11\pi}{48}.$$

(b) A shell has radius x and height $\frac{1}{\sqrt{x}}-1$.

The volume is $\int_{1/4}^1 2\pi(x)\left(\frac{1}{\sqrt{x}}-1\right)dx$

$$=2\pi\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2\right]_{1/4}^1$$
$$=\frac{11\pi}{48}.$$

48. **(a)** For $0<x\le\pi$, $x\,f(x)=\frac{x(\sin x)}{x}=\sin x$.

For $x=0$, $xf(x)=0\cdot 1=\sin 0=\sin x$, so $xf(x)=\sin x$ for $0\le x\le\pi$.

(b) Use cylindrical shells: a shell has radius x and height y. The volume is $\int_0^\pi 2\pi xy\,dx$, which from part (a) is

$$\int_0^\pi 2\pi\sin x\,dx=2\pi[-\cos x]_0^\pi=4\pi.$$

49. **(a)** A cross section has radius $r=\frac{x}{12}\sqrt{36-x^2}$ and area $A(x)=\pi r^2=\frac{\pi}{144}(36x^2-x^4)$. The volume is $\int_0^6\frac{\pi}{144}(36x^2-x^4)\,dx$

$$=\frac{\pi}{144}\left[12x^3-\frac{1}{5}x^5\right]_0^6$$
$$=\frac{36\pi}{5}\text{cm}^3.$$

(b) $\left(\frac{36\pi}{5}\text{cm}^3\right)(8.5\,\text{g}/\text{cm}^3)\approx 192.3\,\text{g}.$

50. **(a)** A cross section has radius $r=\sqrt{2y}$ and area $\pi r^2=2\pi y$. The volume is

$$\int_0^5 2\pi y\,dy=\pi[y^2]_0^5=25\pi.$$

(b) $V(h)=\int A(h)\,dh$, so $\frac{dV}{dh}=A(h)$.

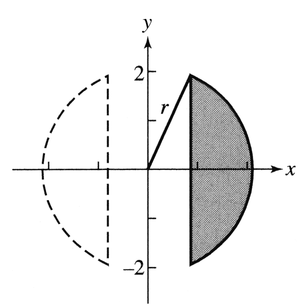
$$\therefore \frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}=A(h)\cdot\frac{dh}{dt},$$

so $\frac{dh}{dt}=\frac{1}{A(h)}\cdot\frac{dV}{dt}$

For $h=4$, the area is $2\pi(4)=8\pi$,

so $\frac{dh}{dt}=\frac{1}{8\pi}\cdot 3\frac{\text{units}^3}{\text{sec}}=\frac{3}{8\pi}\cdot\frac{\text{units}^3}{\text{sec}}.$

51. **(a)**



The remaining solid is that swept out by the shaded region in revolution. Use cylindrical shells: a shell has radius x and

height $2\sqrt{r^2-x^2}$. The volume is

$$\int_{\sqrt{r^2-2^2}}^{r} 2\pi(x)\left(2\sqrt{r^2-x^2}\right)dx$$

$$=2\pi\left[-\frac{2}{3}(r^2-x^2)^{3/2}\right]_{\sqrt{r^2-4}}^{r}$$

$$=-\frac{4}{3}\pi(-8)$$

$$=\frac{32\pi}{3}.$$

(b) The answer is independent of *r*.

52. Partition the appropriate interval in the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\,dx$.

53. Solve $ax-x^2=0$: This is true at $x=a$. For revolution about the *x*-axis, a cross section has radius $r=ax-x^2$ and area

$$A(x)=\pi r^2$$
$$=\pi(ax-x^2)^2$$
$$=\pi(a^2x^2-2ax^3+x^4).$$

The volume is

$$\int_0^a \pi(a^2x^2-2ax^3+x^4)\,dx$$

$$=\pi\left[\frac{1}{3}a^2x^3-\frac{1}{2}ax^4+\frac{1}{5}x^5\right]_0^a$$

$$=\frac{1}{30}\pi a^5.$$

For revolution about the *y*-axis, a cylindrical shell has radius *x* and height $ax-x^2$. The volume is

$$\int_0^a 2\pi(x)(ax-x^2)\,dx=2\pi\left[\frac{1}{3}ax^3-\frac{1}{4}x^4\right]_0^a$$

$$=\frac{1}{6}\pi a^4.$$

Setting the two volumes equal,

$\frac{1}{30}\pi a^5=\frac{1}{6}\pi a^4$ yields $\frac{1}{30}a=\frac{1}{6}$, so $a=5$.

54. The slant height Δs of a tiny horizontal slice can be written as $\Delta s=\sqrt{\Delta x^2+\Delta y^2}=\sqrt{1+(g'(y))^2}\,\Delta y$. So the surface area is approximated by the Riemann sum $\sum_{k=1}^{n} 2\pi g(y_k)\sqrt{1+(g'(y))^2}\,\Delta y$. The limit of that is the integral.

55. $g'(y)=\frac{dx}{dy}=\frac{1}{2\sqrt{y}}$, and

$$\int_0^2 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy=\int_0^2 \pi\sqrt{4y+1}\,dy$$

$$=\left[\frac{\pi}{6}(4y+1)^{3/2}\right]_0^2$$

$$=\frac{13\pi}{3}\approx 13.614$$

56. $g'(y)=\frac{dx}{dy}=y^2$, and

$$\int_0^1 2\pi\left(\frac{y^3}{3}\right)\sqrt{1+(y^2)^2}\,dy$$

$$=\frac{2}{3}\pi\left[\frac{1}{6}(1+y^4)^{3/2}\right]_0^1$$

$$=\frac{\pi}{9}\left(2\sqrt{2}-1\right)\approx 0.638.$$

57. $g'(y)=\frac{dx}{dy}=\frac{1}{2}y^{-1/2}$, and

$$\int_1^3 2\pi\left[y^{1/2}-\left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1+\left(\frac{1}{2}y^{-1/2}\right)^2}\,dy$$

$$=2\pi\int_1^3\left[y^{1/2}-\left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1+\frac{1}{4y}}\,dy.$$

Using NINT, this evaluates to ≈ 16.110

58. $g'(y)=\frac{dx}{dy}=\frac{1}{\sqrt{2y-1}}$, and

$$\int_{5/8}^1 2\pi\sqrt{2y-1}\sqrt{1+\left(\frac{1}{2y-1}\right)^2}\,dy$$

$$=2\pi\int_{5/8}^1 \sqrt{2y}\,dy$$

$$=2\sqrt{2}\pi\left[\frac{2}{3}y^{3/2}\right]_{5/8}^1$$

$$=\frac{4\sqrt{2}}{3}\pi\left(1-\frac{5}{16}\sqrt{\frac{5}{2}}\right)\approx 2.997.$$

59. $f'(x) = \frac{dy}{dx} = 2x$, and

$\int_0^2 2\pi x^2 \sqrt{1+(2x)^2}\, dx = \int_0^2 2\pi x^2 \sqrt{1+4x^2}\, dx$ evaluates, using NINT, to ≈ 53.226.

60. $f'(x) = \frac{dy}{dx} = 3-2x$, and

$\int_0^3 2\pi(3x-x^2)\sqrt{1+(3x-2x)^2}\, dx$ evaluates, using NINT, to ≈ 44.877.

61. $f'(x) = \frac{dy}{dx} = \frac{1-x}{\sqrt{2x-x^2}}$, and

$$\int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\, dx$$
$$= 2\pi\int_{0.5}^{1.5} 1\, dx$$
$$= 2\pi[x]_{0.5}^{1.5}$$
$$= 2\pi \approx 6.283$$

62. $f'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}$, and

$$\int_1^5 2\pi\sqrt{x+1}\sqrt{1+\left(\frac{1}{2\sqrt{x+1}}\right)^2}\, dx$$
$$= 2\pi\int_1^5 \sqrt{x+\frac{5}{4}}\, dx$$
$$= 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$$
$$= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right]$$
$$= \frac{49\pi}{3} \approx 51.313$$

63. True; by definition

64. False; the volume is given by $\int_0^2 \pi y^4\, dy$.

65. A; $V = \int_1^e (\ln(x))^2\, dx = 0.718$

66. E; $V = \int_0^4 (\pi(8-x^{3/2})^2)\, dx = 361.9$

67. B; $V = \int_0^{16} \pi\left(4^2 - \left(\sqrt{y}\right)^2\right) dy = 128\pi$

68. D

69. A cross section has radius $r = |c - \sin x|$ and area $A(x) = \pi r^2$

$$= \pi(c-\sin x)^2$$
$$= \pi(c^2 - 2c\sin x + \sin^2 x).$$

The volume is

$$\int_0^\pi \pi(c^2 - 2c\sin x + \sin^2 x)\, dx$$
$$= \pi\left[c^2 x - 2c\cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi$$
$$= \pi\left[\left(c^2\pi - 2c + \frac{1}{2}\pi\right) - 2c\right]$$
$$= \pi^2 c^2 - 4\pi c + \frac{\pi^2}{2}.$$

(a) Solve

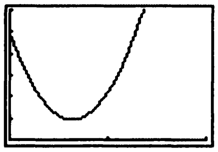
$$\frac{d}{dc}\left[\pi^2 c^2 - 4\pi c + \frac{\pi^2}{2}\right] = 0$$
$$2\pi^2 c - 4\pi = 0$$
$$\pi c - 2 = 0$$
$$c = \frac{2}{\pi}$$

This value of c gives a minimum for V because $\frac{a^2 V}{dc^2} = 2\pi^2 > 0$.

Then the volume is

$$\pi^2\left(\frac{2}{\pi}\right)^2 - 4\pi\left(\frac{2}{\pi}\right) + \frac{\pi^2}{2} = \frac{\pi^2}{2} - 4$$

(b) Since the derivative with respect to c is not zero anywhere else besides $c = \frac{2}{\pi}$, the maximum must occur at $c = 0$ or $c = 1$. The volume for $c = 0$ is $\frac{\pi^2}{2} \approx 4.935$, and for $c = 1$ it is $\frac{\pi(3\pi-8)}{2} \approx 2.238$. $c = 0$ maximizes the volume.

(c) 

[0, 2] by [0, 6]

The volume gets large without limit. This makes sense, since the curve is sweeping out space in an ever-increasing radius.

70. **(a)** Using $d = \dfrac{C}{\pi}$, and $A = \pi\left(\dfrac{d}{2}\right)^2 = \dfrac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth): 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2.

(b) If $C(y)$ is the circumference as a function of y, then the area of a cross section is

$$A(y) = \pi\left(\frac{\frac{C(y)}{\pi}}{2}\right)^2 = \frac{(C(y))^2}{4\pi}, \text{ and the volume is } \frac{1}{4\pi}\int_0^6 (C(y))^2\,dy.$$

(c)
$$\begin{aligned}\frac{1}{4\pi}\int_0^6 A(y)\,dy &= \frac{1}{4\pi}\int_0^6 \big(C(y)\big)^2\,dy\\ &\approx \frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2 + 2(4.5^2 + 4.4^2 + 5.1^2 + 6.3^2 + 7.8^2 + 9.4^2 + 10.8^2 + 11.6^2 + 11.6^2\\ &\qquad + 10.8^2 + 9.0^2) + 6.3^2]\\ &\approx 34.7 \text{ in}^3\end{aligned}$$

71. Hemisphere cross sectional area: $\pi\left(\sqrt{R^2 - h^2}\right)^2 = A_1$.

Right circular cylinder with cone removed cross sectional area: $\pi R^2 - \pi h^2 = A_2$
Since $A_1 = A_2$, the two volumes are equal by Cavalieri's theorem. Thus
volume of hemisphere
= volume of cylinder – volume of cone
$= \pi R^3 - \dfrac{1}{3}\pi R^3$
$= \dfrac{2}{3}\pi R^3.$

72. Use washer cross sections: a washer has inner radius $r = b - \sqrt{a^2 - y^2}$, outer radius $R = b + \sqrt{a^2 + y^2}$, and area $\pi(R^2 - r^2) = \pi\left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right]$

$\pi\int_0^2 \left(\sqrt{2y}\right)^2\,dy = \pi\int_0^2 2y\,dy = \pi y^2\Big]_0^2 = 4\pi.$ The volume is

$$\begin{aligned}\int_{-a}^{a} 4\pi b\sqrt{a^2 - y^2}\,dy &= 4\pi b\int_{-a}^{a}\sqrt{a^2 - y^2}\,dy\\ &= 4\pi b\left(\frac{\pi a^2}{2}\right)\\ &= 2\pi^2 a^2 b\end{aligned}$$

73. **(a)** Put the bottom of the bowl at (0, –a). The area of a horizontal cross section is

$$\pi\left(\sqrt{a^2-y^2}\right)^2=\pi(a^2-y^2).$$

The volume for height h is

$$\int_{-a}^{h-a}\pi(a^2-y^2)\,dy=\pi\left[a^2y-\frac{1}{3}y^3\right]_{-a}^{h-a}$$
$$=\frac{\pi h^2(3a-h)}{3}.$$

(b) For $h=4$, $y=-1$ and the area of a cross section is $\pi(5^2-1^2)=24\pi$. The rate of rise is

$$\frac{dh}{dt}=\frac{1}{A}\frac{dV}{dt}=\frac{1}{24\pi}(0.2)=\frac{1}{120\pi}\text{ m/sec.}$$

74. **(a)** A cross section has radius $r=\sqrt{a^2-x^2}$ and area $A(x)=\pi r^2$

$$=\pi\left(\sqrt{a^2-x^2}\right)^2$$
$$=\pi(a^2-x^2).$$

The volume is

$$\int_{-a}^{a}\pi(a^2-x^2)dx$$
$$=\pi\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{a}$$
$$=\pi\left[\left(a^3-\frac{1}{3}a^3\right)-\left(-a^3+\frac{1}{3}a^3\right)\right]$$
$$=\frac{4}{3}\pi a^3.$$

(b) A cross section has radius $x=r\left(1-\frac{y}{h}\right)$

and area $A(y)=\pi x^2$

$$=\pi r^2\left(1-\frac{y}{h}\right)^2$$
$$=\pi r^2\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right).$$

The volume is

$$\int_0^h\pi r^2\left(1-\frac{2y}{h}+\frac{y^2}{h}\right)dy$$
$$=\pi r^2\left[y-\frac{y^2}{h}+\frac{y^3}{3h^2}\right]_0^h$$
$$=\frac{1}{3}\pi r^2h.$$

Quick Quiz Sections 8.1–8.3

1. C; $\int_0^1(\sin^{-1}(x))^2\,dx=0.467$

2. A

3. D

4. **(a)** The two graphs intersect where $\sqrt{x}=e^{-x}$, which a calculator shows to be $x=0.42630275$. Store this value as A. The area of R is

$$\int_0^A\left(e^{-x}-\sqrt{x}\right)dx=0.162.$$

(b) Volume

$$=\int_0^A\pi\left(\left(e^{-x}+1\right)^2-\left(\sqrt{x}+1\right)^2\right)dx$$
$$=1.631.$$

(c) Volume $=\int_0^A\frac{1}{2}\pi\left(\frac{e^{-x}-\sqrt{x}}{2}\right)^2dx$

$$=0.035.$$

Section 8.4 Lengths of Curves (pp. 416–422)

Quick Review 8.4

1. $\sqrt{1+2x+x^2}=\sqrt{(1+x)^2}$, which, since $x\ge-1$, equals $1+x$ or $x+1$.

2. $\sqrt{1-x+\frac{x^2}{4}}=\sqrt{\left(1-\frac{x}{2}\right)^2}$, which, since $x\le 2$, equals $1-\frac{x}{2}$ or $\frac{2-x}{2}$.

3. $\sqrt{1+(\tan x)^2}=\sqrt{(\sec x)^2}$, which since $0\le x<\frac{\pi}{2}$, equals $\sec x$.

4. $\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}=\sqrt{\frac{1}{2}+\frac{1}{16}x^2+\frac{1}{x^2}}$

$$=\frac{1}{4}\sqrt{\frac{(x^2+4)^2}{x^2}}$$

which, since $x>0$, equals $\frac{x^2+4}{4x}$.

5. $\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x},$ which since $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2},$ equals $\sqrt{2}\cos x.$

6. $f(x)$ has a corner at $x = 4.$

7. $\dfrac{d}{dx}(5x^{2/3}) = \dfrac{10}{3\sqrt[3]{x}}$ is undefined at $x = 0.$
$f(x)$ has a cusp there.

8. $\dfrac{d}{dx}\left(\sqrt[5]{x+3}\right) = \dfrac{1}{5(x+3)^{4/5}}$ is undefined for $x = -3.$ $f(x)$ has a vertical tangent there.

9. $\sqrt{x^2 - 4x + 4} = |x-2|$ has a corner at $x = 2.$

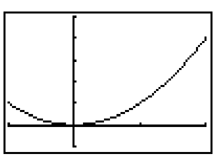
10. $\dfrac{d}{dx}\left(1+\sqrt[3]{\sin x}\right) = \dfrac{\cos x}{3(\sin x)^{2/3}}$ is undefined for $x = k\pi,$ where k is any integer. $f(x)$ has vertical tangents at these values of $x.$

Section 8.4 Exercises

1. (a) $y' = 2x,$ so

$$\text{Length } = \int_{-1}^{2} \sqrt{1+(2x)^2}\,dx = \int_{-1}^{2} \sqrt{1+4x^2}\,dx.$$

(b)



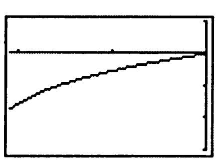
$[-1, 2]$ by $[-1, 5]$

(c) Length ≈ 6.126

2. (a) $y' = \sec^2 x,$ so

$$\text{Length } = \int_{-\pi/3}^{0} \sqrt{1+\sec^4 x}\,dx.$$

(b)

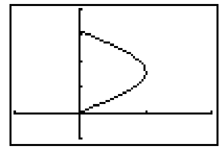


$\left[-\frac{\pi}{3}, 0\right]$ by $[-3, 1]$

(c) Length ≈ 2.057

3. (a) $x' = \cos y,$ so Length $= \int_0^{\pi} \sqrt{1+\cos^2 y}\,dy.$

(b)



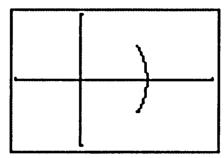
$[-1, 2]$ by $[-1, 4]$

(c) Length ≈ 3.820

4. (a) $x' = -y(1-y^2)^{-1/2},$ so

$$\text{Length } = \int_{-1/2}^{1/2} \sqrt{1+\frac{y^2}{1-y^2}}\,dy.$$

(b)



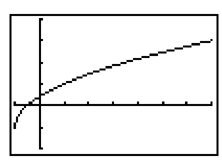
$[-1, 2]$ by $[-1, 1]$

(c) Length ≈ 1.047

5. (a) $y^2 + 2y = 2x + 1,$ so
$y^2 + 2y + 1 = (y+1)^2 = 2x + 2,$ and
$y = \sqrt{2x+2} - 1.$ Then $y' = \dfrac{1}{\sqrt{2x+2}},$ but
NINT may fail using over the entire interval because y' is undefined at $x = -1.$

So, use $x = \dfrac{(y+1)^2}{2} - 1.$ Then $x' = y + 1$
and Length $= \int_{-1}^{3} \sqrt{1+(y+1)^2}\,dy.$

(b)



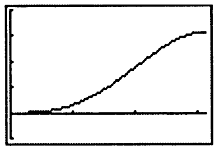
$[-1, 7]$ by $[-2, 4]$

(c) Length ≈ 9.294

6. (a) $y' = \cos x + x\sin x - \cos x = x\sin x,$ so

$$\text{Length } = \int_0^{\pi} \sqrt{1+x^2\sin^2 x}\,dx.$$

(b)



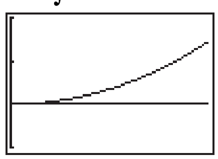
$[0, \pi]$ by $[-1, 4]$

(c) Length ≈ 4.698

7. **(a)** $y' = \tan x,$ so

$$\text{Length } = \int_0^{\pi/6} \sqrt{1+\tan^2 x}\, dx.$$

(b) $y = \int \tan x\, dx = \ln(\sec x)$



$\left[0, \frac{\pi}{6}\right]$ by $[-0.1, 0.2]$

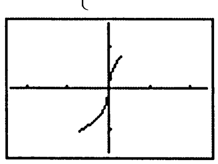
(c) Length ≈ 0.549

8. **(a)** $x' = \sqrt{\sec^2 y - 1},$ so

$$\text{Length} = \int_{-\pi/3}^{\pi/4} \sec y dy.$$

(b) $x' = \sqrt{\sec^2 y - 1} = |\tan y|,$

$$\text{so } x = \begin{cases} \ln(\cos y), & -\frac{\pi}{3} \le y \le 0 \\ -\ln(\cos y), & 0 < y \le \frac{\pi}{4} \end{cases}$$



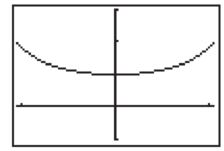
$[-2.4, 2.4]$ by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) Length ≈ 2.198

9. **(a)** $y' = \sec x \tan x,$ so

$$\text{Length} = \int_{-\pi/3}^{\pi/3} \sqrt{1+\sec^2 x \tan^2 x}\, dx.$$

(b)



$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-1, 3]$

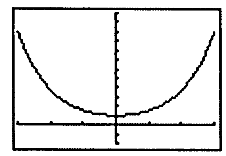
(c) Length ≈ 3.139

10. **(a)** $dx = t^3$

$dy = t^3\, dt$

$dy = 3t^2$

(b)



$[-3, 3]$ by $[-2, 12]$

(c) Length ≈ 20.036

11. $y' = \frac{1}{2}(x^2+2)^{1/2}(2x) = x\sqrt{x^2+2},$ so the length is

$$\int_0^3 \sqrt{1+\left(x\sqrt{x^2+2}\right)^2}\, dx = \int_0^3 \sqrt{x^4+2x^2+1}\, dx$$
$$= \int_0^3 (x^2+1)\, dx$$
$$= \left[\frac{1}{3}x^3 + x\right]_0^3$$
$$= 12.$$

12. $y' = \frac{3}{2}\sqrt{x},$ so the length is

$$\int_0^4 \sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}\, dx = \int_0^4 \sqrt{1+\frac{9x}{4}}\, dx$$
$$= \left[\frac{8}{27}\left(1+\frac{9x}{4}\right)^{3/2}\right]_0^4$$
$$= \frac{80\sqrt{10}-8}{27}.$$

13. $x' = y^2 - \dfrac{1}{4y^2}$, so the length is

$$\int_1^3 \sqrt{1+\left(y^2 - \frac{1}{4y^2}\right)^2}\, dy$$
$$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\, dy$$
$$= \left[\frac{1}{3}y^3 - \frac{1}{4y}\right]_1^3$$
$$= \frac{53}{6}.$$

14. $x' = y^3 - \dfrac{1}{4y^3}$, so the length is

$$\int_1^2 \sqrt{1+\left(y^3 - \frac{1}{4y^3}\right)^2}\, dy = \int_1^2 \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\, dy$$
$$= \left[\frac{1}{4}y^4 - \frac{1}{8y^2}\right]_1^2$$
$$= \frac{123}{32}.$$

15. $x' = \dfrac{y^2}{2} - \dfrac{1}{2y^2}$, so the length is

$$\int_1^2 \sqrt{1+\left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\, dy$$
$$= \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}\, dy$$
$$= \left[\frac{1}{6}y^3 - \frac{1}{2y}\right]_1^2$$
$$= \frac{17}{12}.$$

16. $y' = x^2 + 2x + 1 - \dfrac{4}{(4x+4)^2}$

$$= (x+1)^2 - \frac{1}{4(x+1)^2}$$

so the length is

$$\int_0^2 \sqrt{1+\left((x+1)^2 - \frac{1}{4(x+1)^2}\right)^2}\, dx$$
$$= \int_0^2 \sqrt{\left((x+1)^2 + \frac{1}{4(x+1)^2}\right)^2}\, dx$$
$$= \left[\frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)}\right]_0^2$$
$$= \frac{53}{6}.$$

17. $x' = \sqrt{\sec^4 y - 1}$, so the length is

$$\int_{-\pi/4}^{\pi/4} \sqrt{1+(\sec^4 y - 1)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy$$
$$= [\tan y]_{-\pi/4}^{\pi/4}$$
$$= 2.$$

18. $y' = \sqrt{3x^4 - 1}$, so the length is

$$\int_{-2}^{-1} \sqrt{1+(3x^4-1)}\, dx = \int_{-2}^{-1} \sqrt{3}x^2\, dx$$
$$= \sqrt{3}\left[\frac{1}{3}x^3\right]_{-2}^{-1}$$
$$= \frac{7\sqrt{3}}{3}.$$

19. (a) $\left(\dfrac{dy}{dx}\right)^2$ corresponds to $\dfrac{1}{4x}$ here, so take $\dfrac{dy}{dx}$ as $\dfrac{1}{2\sqrt{x}}$.

Then $y = \sqrt{x} + C$, and, since (1, 1) lies on the curve, $C = 0$. So $y = \sqrt{x}$.

(b) Two; we know the value of the function at one value of x, and we know the square of the derivative. We can also let $\dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{x}}$, which gives the solution $y = 2 - \sqrt{x}$.

20. **(a)** $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dx}{dy}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since $(0, 1)$ lies on the curve, $C = 1$. So

$$y = \frac{1}{1-x}.$$

(b) Two; we know the value of the function at one value of x, and we know the square of the derivative. We can also let $\frac{dx}{dy} = -\frac{1}{y^2}$, which gives the solution

$$y = \frac{1}{x+1}.$$

21. $y' = \sqrt{\cos 2x}$, so the length is

$$\int_0^{\pi/4} \sqrt{1+\cos 2x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx$$
$$= \sqrt{2}[\sin x]_0^{\pi/4}$$
$$= 1.$$

22. $y' = -(1-x^{2/3})^{1/2} x^{-1/3}$, so the length is

$$8\int_{\sqrt{2}/4}^{1} \sqrt{1+(1-x^{2/3})\, x^{-2/3}}\, dx$$
$$= 8\int_{\sqrt{2}/4}^{1} \sqrt{x^{-2/3}}\, dx$$
$$= 8\int_{\sqrt{2}/4}^{1} x^{-1/3} dx$$
$$= 8\left[\frac{3}{2}x^{2/3}\right]_{\sqrt{2}/4}^{1}$$
$$= 8\left[\frac{3}{2} - \frac{3}{2}\left(\frac{1}{2}\right)\right]$$
$$= 6.$$

23. Find the length of the curve

$y = \sin\frac{3\pi}{20}x$ for $0 \le x \le 20$.

$y' = \frac{3\pi}{20}\cos\frac{3\pi}{20}x$, so the length is

$\int_0^{20} \sqrt{1+\left(\frac{3\pi}{20}\cos\frac{3\pi}{20}x\right)^2}\, dx$, which evaluates, using NINT, to ≈ 21.07 inches.

24. The area is 300 times the length of the arch.

$y' = -\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi x}{50}\right)$, so the length is

$\int_{-25}^{25} \sqrt{1+\left(\frac{\pi}{2}\right)^2 \sin^2\left(\frac{\pi x}{50}\right)}\, dx$, which evaluates, using NINT, to ≈ 73.185. Multiply that by 300, then by \$1.75 to obtain the cost (rounded to the nearest dollar): \$38,422.

25. $f'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-1/3}$, but NINT fails on $\int_0^2 \sqrt{1+(f'(x))^2}\, dx$ because of the undefined slope at $x = 0$. So, instead solve for x in terms of y using the quadratic formula.

$(x^{1/3})^2 + x^{1/3} - y = 0$, and

$x^{1/3} = \frac{-1 \pm \sqrt{1+4y}}{2}$. Using the positive values, $x = \frac{1}{8}\left(\sqrt{1+4y} - 1\right)^3$. Then,

$x' = \frac{3}{8}\left(\sqrt{1+4y} - 1\right)^2 \left(\frac{2}{\sqrt{1+4y}}\right)$, and

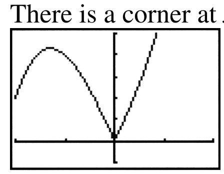
$$\int_0^{2^{1/3}+2^{2/3}} \sqrt{1+(x')^2}\, dy \approx 3.6142.$$

26. $$f'(x) = \frac{(4x^2+1)-(8x^2-8x)}{(4x^2+1)^2}$$
$$= -\frac{4x^2-8x-1}{(4x+1)^2},$$

so the length is $\int_{-1/2}^{1} \sqrt{1+\left(\frac{4x^2-8x-1}{(4x^2+1)^2}\right)^2}\, dx$

which evaluates, using NINT, to ≈ 2.1089.

27. There is a corner at $x = 0$:



[–2, 2] by [–1, 5]

Break the function into two smooth segments:

$y = \begin{cases} x^3 - 5x, & -2 \le x \le 0 \\ x^3 + 5x, & 0 < x \le 1 \end{cases}$ and

$y' = \begin{cases} 3x^2 - 5, & -2 \le x < 0 \\ 3x^2 + 5, & 0 < x \le 1 \end{cases}$

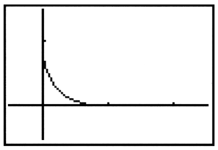
The length is

$\int_{-2}^{1} \sqrt{1+(y')^2}\, dy$

$= \int_{-2}^{0} \sqrt{1+(3x^2-5)^2}\, dx + \int_{0}^{1} \sqrt{1+(3x^2+5)^2}\, dx,$

which evaluates, using NINT for each part, to ≈ 13.132.

28. $y = (1-\sqrt{x})^2,\ 0 \le x \le 1$

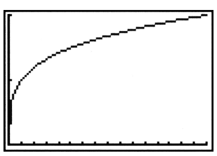


[–0.5, 2.5] by [–0.5, 1.5]

$y' = \dfrac{\sqrt{x}-1}{\sqrt{x}}$, but NINT may fail using y' over the entire interval because y' is undefined at $x = 0$. So, split the curve into two equal segments by solving $\sqrt{x} + \sqrt{y} = 1$ with $y = x$: $x = \frac{1}{4}$. The total length is

$2\int_{1/4}^{1} \sqrt{1+\left(\dfrac{\sqrt{x}-1}{\sqrt{x}}\right)^2}\, dx,$ which evaluates, using NINT, to ≈ 1.623.

29.



[0, 16] by [0, 2]

$y' = \frac{1}{4}x^{-3/4}$, but NINT may fail using y' over the entire interval, because y' is undefined at $x = 0$. So, use $x = y^4,\ 0 \le y \le 2$: $x' = 4y^3$ and the length is $\int_0^2 \sqrt{1+(4y^3)^2}\, dy$, which evaluates, using NINT, to ≈ 16.647.

30. Horizontal line segments will not work because the limit of the sum $\sum \Delta x_k$, as the norm of the partition goes to zero, will always be the length ($b - a$) of the interval (a, b).

31. No; the curve can be indefinitely long. Consider, for example, the curve $\frac{1}{3}\sin\left(\frac{1}{x}\right) + 0.5$ for $0 < x < 1$.

32. False; the function must be differentiable.

33. True, by definition.

34. D; $\dfrac{dy}{dx} = -2\sin(2x)$

$\int_0^{\pi/4} \sqrt{1+(-2\sin(2x))^2}\, dx \approx 1.318$

35. C; $\dfrac{dx}{dy} = 3y^2$

$\int_{-2}^{2} \sqrt{1+(3y^2)^2}\, dy = \int_{-2}^{2} \sqrt{1+9y^4}\, dy$

36. B; $\dfrac{dy}{dx} = \sqrt{x}$

$\int_0^8 \sqrt{1+\sqrt{x}^2}\, dx = \dfrac{52}{3}$

37. A; $\dfrac{dx}{dy} = \dfrac{3}{2}y^{1/2}$

$2\int_0^1 \sqrt{1+\left(\dfrac{3}{2}y^{1/2}\right)^2}\, dy = 2\int_0^1 \sqrt{1+\dfrac{9}{4}y}\, dy$

38. For track 1: $y_1 = 0$ at $x = \pm 10\sqrt{5} \approx \pm 22.3607$, and $y_1{}' = \dfrac{-0.2x}{\sqrt{100-0.2x^2}}$. NINT fails to evaluate $\int_{-10\sqrt{5}}^{10\sqrt{5}} \sqrt{1+(y_1{}')^2}\, dx$ because of the undefined slope at the limits, so use the track's symmetry, and "back away" from the upper limit a little, and find $2\int_0^{22.36} \sqrt{1+(y_1{}')^2}\, dx \approx 52.548$. Then, approximating the last little stretch at each end by a straight vertical line, add $2\sqrt{100-0.2(22.36)^2} \approx 0.156$ to get the total length of track 1 as ≈ 52.704. Using a similar strategy, find the length of the *right half* of track 2 to be ≈ 32.274. Store the unrounded value as A. Now enter $Y_1 = 52.704$ and

Y_2

$= \text{A} + \text{NINT}\left(\sqrt{1+\left(\dfrac{-0.2t}{\sqrt{150-0.2t^2}}\right)^2},\ t,\ x,\ 0\right)$

and graph in a [–30, 0] by [0, 60] window to see the effect of the x-coordinate of the lane-2 starting position on the length of lane 2. (Be patient!) Solve graphically to find the intersection at $x \approx -19.909$, which leads to starting point coordinates (–19.909, 8.410).

39. **(a)** The fin is the hypotenuse of a right triangle with leg lengths Δx_k and $\left.\frac{df}{dx}\right|_{x=x_{k-1}} \Delta x_k = f'(x_{k-1})\Delta x_k$.

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x_k \sqrt{1+(f'(x_{k-1}))^2}$

$= \int_a^b \sqrt{1+(f'(x))^2}\, dx$

40. Yes; any curve of the form $y = \pm x + c$, where c is a constant, has constant slope ± 1, so that $\int_0^a \sqrt{1+(y')^2}\, dx = \int_0^a \sqrt{2}\, dx = a\sqrt{2}$.

Section 8.5 Applications from Science and Statistics (pp. 423–433)

Quick Review 8.5

1. **(a)** $\int_0^1 e^{-x}\, dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}$

(b) ≈ 0.632

2. **(a)** $\int_0^1 e^x\, dx = [e^x]_0^1 = e - 1$

(b) ≈ 1.718

3. **(a)** $\int_{\pi/4}^{\pi/2} \sin x\, dx = [-\cos x]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

(b) ≈ 0.707

4. **(a)** $\int_0^3 (x^2+2)\, dx = \left[\frac{1}{3}x^3 + 2x\right]_0^3 = 15$

(b) 15

5. **(a)** $\int_1^2 \frac{x^2}{x^3+1}\, dx = \left[\frac{1}{3}\ln(x^3+1)\right]_1^2$

$= \frac{1}{3}[\ln 9 - \ln 2]$

$= \frac{1}{3}\ln\left(\frac{9}{2}\right)$

(b) ≈ 0.501

6. $\int_0^7 2\pi(x+2)\sin x\, dx$

7. $\int_0^7 (1-x^2)(2\pi x)\, dx$

8. $\int_0^7 \pi \cos^2 x\, dx$

9. $\int_0^7 \pi\left(\frac{y}{2}\right)^2 (10-y)\, dy$

10. $\int_0^7 \frac{\sqrt{3}}{4}\sin^2 x\, dx$

Section 8.5 Exercises

1. $\int_0^5 xe^{-x/3}\, dx = [-3e^{-x/3}(3+x)]_0^5$

$= -\frac{24}{e^{5/3}} + 9 \approx 4.4670$ J

2. $\int_0^3 x\sin\left(\frac{\pi x}{4}\right) dx$

$= \frac{4}{\pi}\left[\frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right) - x\cos\left(\frac{\pi x}{4}\right)\right]_0^3$

$= \frac{4\sqrt{2}}{\pi}\left(\frac{2}{\pi} + \frac{3}{2}\right) \approx 3.8473$ J

3. $\int_0^3 x\sqrt{9-x^2}\, dx = \left[-\frac{1}{3}(9-x^2)^{3/2}\right]_0^3 = 9$ J

4. $\int_0^{10} (e^{\sin x}\cos x + 2)\, dx = [e^{\sin x} + 2x]_0^{10}$

$= e^{\sin 10} + 19$

≈ 19.5804 J

5. When the bucket is x m off the ground, the water weighs

$F(x) = 490\left(\frac{20-x}{20}\right)$

$= 490\left(1 - \frac{x}{20}\right)$

$= 490 - 24.5x$ N

Then

$W = \int_0^{20} (490 - 24.5x)\, dx$

$= [490x - 12.25x^2]_0^{20}$

$= 4900$ J.

6. When the bucket is x m off the ground, the water weighs

$$F(x) = 490\left(\frac{20-\frac{4x}{5}}{20}\right)$$
$$= 490\left(1-\frac{x}{25}\right)$$
$$= 490 - 19.6x \text{ N.}$$

Then

$$W = \int_0^{20}(490-19.6x)\,dx$$
$$= [490x - 9.8x^2]_0^{20}$$
$$= 5880 \text{ J.}$$

7. When the bag is x ft off the ground, the sand weighs

$$F(x) = 144\left(\frac{18-\frac{x}{2}}{18}\right)$$
$$= 144\left(1-\frac{x}{36}\right)$$
$$= 144 - 4x \text{ lb}$$

Then $W = \int_0^{18}(144-4x)\,dx$

$$= [144x - 2x^2]_0^{18}$$
$$= 1944 \text{ ft-lb}$$

8. **(a)** $F = ks$, so $800 = k(14 - 10)$ and $k = 200$ lb/in.

(b) $F(x) = 200x$, and

$$\int_0^2 200x\,dx = [100x^2]_0^2 = 400 \text{ in-lb.}$$

(c) $F = 200s$, so $s = \frac{1600}{200} = 8$ in.

9. **(a)** $F = ks$, so $21{,}714 = k(8 - 5)$ and $k = 7238$ lb/in.

(b) $F(x) = 7238x$

$$W = \int_0^{1/2} 7238x\,dx$$
$$= [3619x^2]_0^{1/2}$$
$$= 904.75 \approx 905 \text{ in-lb,}$$

and $W = \int_{1/2}^{1} 7238x\,dx$

$$= [3619x^2]_{1/2}^{1}$$
$$= 2714.25 \approx 2714 \text{ in.-lb}$$

10. **(a)** $F = ks$, so $150 = k\left(\frac{1}{16}\right)$ and $k = 2400$ lb/in. Then for $s = \frac{1}{8}$,

$$F = 2400\left(\frac{1}{8}\right) = 300 \text{ lb.}$$

(b) $\int_0^{1/8} 2400x\,dx = [1200x^2]_0^{1/8}$

$$= 18.75 \text{ in.-lb}$$

11. When the end of the rope is x m from its starting point, the $(50 - x)$ m of rope still to go weigh $F(x) = (0.624)(50 - x)$ N. The total work is $\int_0^{50}(0.624)(50-x)\,dx$

$$= 0.624\left[50x - \frac{1}{2}x^2\right]_0^{50}$$
$$= 780 \text{ J}$$

12. **(a)** Work $\int_{(p_1,V_1)}^{(p_2,V_2)} F(x)\,dx = \int_{(p_1,V_1)}^{(p_2,V_2)} pA\,dx$

$$= \int_{(p_1,V_1)}^{(p_2,V_2)} p\,dV$$

(b) $p_1V_1^{1.4} = (50)(243)^{1.4} = 109{,}350,$ so

$p = \frac{109{,}350}{V^{1.4}}$ and

$$\text{Work } = \int_{(p_1,V_1)}^{(p_2,V_2)} \frac{109{,}350}{V^{1.4}}\,dV$$
$$= 109{,}350[-2.5V^{-0.4}]_{V=243}^{V=32}$$
$$= -37{,}968.75 \text{ in-lb.}$$

13. **(a)** From the equation $x^2 + y^2 = 3^2$, it follows that a thin horizontal rectangle has area $2\sqrt{9-y^2}\,\Delta y$, where y is distance from the top, and pressure $62.4y$. The total force is approximately

$$\sum_{k=1}^{n}(62.4y_k)\left(2\sqrt{9-y_k^2}\right)\Delta y$$
$$= \sum_{k=1}^{n} 124.8y_k\sqrt{9-y_k^2}\,\Delta y.$$

(b) $\int_0^3 124.8y\sqrt{9-y^2}\,dy$

$$= [-41.6(9-y^2)^{3/2}]_0^3$$
$$= 1123.2 \text{ lb}$$

14. **(a)** From the equation $\frac{x^2}{3^2}+\frac{y^2}{8^2}=1$, it follows that a thin horizontal rectangle has area $6\sqrt{1-\frac{y^2}{64}}\Delta y$, where y is distance from the top, and pressure $62.4y$. The total force is approximately

$$\sum_{k=1}^{n}(62.4y_k)\left(6\sqrt{1-\frac{y_k^2}{64}}\right)\Delta y$$
$$=\sum_{k=1}^{n}374.4y_k\sqrt{1-\frac{y_k^2}{64}}\Delta y.$$

(b) $\int_0^8 374.4y\sqrt{1-\frac{y^2}{64}}\,dy$

$$=\left[-7987.2\left(1-\frac{y^2}{64}\right)^{3/2}\right]_0^8$$
$$=7987.2 \text{ lb}$$

15. **(a)** From the equation $x=\frac{3}{8}y$, it follows that a thin horizontal rectangle has area $\frac{3}{4}y\Delta y$, where y is the distance from the top of the triangle, the pressure is $62.4(y-3)$. The total force is approximately

$$\sum_{k=1}^{n}62.4(y_k-3)\left(\frac{3}{4}y_k\right)\Delta y$$
$$=\sum_{k=1}^{n}46.8(y_k{}^2-3y_k)\Delta y.$$

(b) $\int_3^8 46.8(y_2{}^2-3y)\,dy=[15.6y^3-70.2y^2]_3^8$

$$=3494.4-(-210.6)$$
$$=3705\text{ lb}$$

16. **(a)** From the equation $y=\frac{x^2}{2}$, it follows that a thin horizontal rectangle has area $2\sqrt{2y}\Delta y$ where y is distance from the bottom, and pressure $62.4(4-y)$. The total force is approximately

$$\sum_{k=1}^{n}62.4(4-y_k)\left(2\sqrt{2y_k}\right)\Delta y$$
$$=\sum_{k=1}^{n}124.8\sqrt{2}\left(4\sqrt{y_k}-y_k{}^{3/2}\right)\Delta y.$$

(b) $\int_0^4 124.8\sqrt{2}\left(4\sqrt{y}-y^{3/2}\right)dy$

$$=124.8\sqrt{2}\left[\frac{8}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^4$$
$$=1064.96\sqrt{2}$$

17. **(a)** Work to raise a thin slice $=62.4(10\times 12)(\Delta y)\,y$.

$$\text{Total work}=\int_0^{20}62.4(120y)dy$$
$$=62.4\left[60y^2\right]_0^{20}$$
$$=1,497,600\text{ ft-lb}$$

(b) (1,497,600 ft-lb) ÷ (250 ft-lb/sec)
= 5990.4 sec
≈ 100 min

(c) Work to empty half the tank

$$=\int_0^{10}62.4(120)\,y\,dy$$
$$=62.4\left[60y^2\right]_0^{10}$$
$$=374,400\text{ ft-lb},$$

and 374,400 ÷ 250 = 1497.6 sec ≈ 25 min

(d) The weight per ft^3 of water is a simple multiplicative factor in the answers. So divide by 62.4 and multiply by the appropriate weight-density.
For 62.26:

$$1,497,600\left(\frac{62.26}{62.4}\right)=1,494,240\text{ ft-lb and}$$
$$5990.4\left(\frac{62.26}{62.4}\right)=5976.96\text{ sec}\approx 100\text{ min}.$$

For 62.5:

$$1,497,600\left(\frac{62.5}{62.4}\right)=1,500,000\text{ ft-lb and}$$
$$5990.4\left(\frac{62.5}{62.4}\right)=6000\text{ sec}=100\text{ min}.$$

18. The work needed to raise a thin disk is $\pi(10)^2(51.2)(30-y)\Delta y$, where y is height up from the bottom. The total work is

$$\int_0^{30} 100\pi(51.2)(30-y)\,dy$$
$$=5120\pi\left[30y-\frac{1}{2}y^2\right]_0^{30}$$
$$\approx 7,238,229 \text{ ft-lb}$$

19. Work to pump through the valve is
$\pi(2)^2(62.4)(y+15)\Delta y$
for a thin disk and

$$\int_0^6 4\pi(62.4)(y+15)\,dy = 249.6\pi\left[\frac{1}{2}y^2+15y\right]_0^6$$
$$\approx 84,687.3 \text{ ft-lb}$$

for the whole tank. Work to pump over the rim is $\pi(2)^2(62.4)(6+15)\Delta y$ for a thin disk and

$$\int_0^6 4\pi(62.4)(21)\,dy = 4\pi(62.4)(21)(6)$$
$$\approx 98,801.8 \text{ ft-lb}$$

for the whole tank. Less work is required to fill the tank from the bottom.

20. The work is the same as if the straw were initially an inch long and just touched the surface, and lengthened as the liquid level dropped. For a thin disk, the volume is

$\pi\left(\frac{y+17.5}{14}\right)^2\Delta y$ and the work to raise it is

$\pi\left(\frac{y+17.5}{14}\right)^2\left(\frac{4}{9}\right)(8-y)\Delta y$. The total work is

$\int_0^7 \pi\left(\frac{y+17.5}{14}\right)^2\left(\frac{4}{9}\right)(8-y)\,dy$, which using NINT evaluates to ≈91.3244 in.-oz.

21. The work is that already calculated (to pump the oil to the rim) plus the work needed to raise the entire amount 3 ft higher. The latter comes to

$$\left(\frac{1}{3}\pi r^2 h\right)(57)(3) = 57\pi(4)^2(8) = 22,921 \text{ ft-lb},$$

and the total is
22,921 + 30,561 = 53,482.5 ft-lb.

22. The weight density is a simple multiplicative factor: Divide by 57 and multiply by 64.5.

$$30,561\left(\frac{64.5}{57}\right) \approx 34,582.18 \text{ ft-lb}.$$

23. The work to raise a thin disk is

$$\pi r^2(56)h = \pi\left(\sqrt{10^2-y^2}\right)^2(56)(10+2-y)\Delta y$$
$$= 56\pi(12-y)(100-y^2)\Delta y.$$

The total work is

$\int_0^{10} 56\pi(12-y)(100-y^2)\,dy$, which evaluates using NINT to $\approx 967,611$ ft-1b. This will come to (967,611)($0.005) ≈ $4838, so yes, there's enough money to hire the firm.

24. Pipe radius $=\frac{1}{6}$ ft;

Work to fill pipe $=\int_0^{360}\pi\left(\frac{1}{6}\right)^2(62.4)y\,dy$
$=112,320\pi$ ft-lb.

Work to fill tank $=\int_{360}^{385}\pi(10)^2(62.4)\,y\,dy$
$=58,110,000\pi$ ft-lb

Total work $=58,222,320\pi$ ft-lb, which will take $58{,}222{,}320\pi+1650 \approx 110,855$ sec ≈ 31 hr.

25. **(a)** The pressure at depth y is $62.4y$, and the area of a thin horizontal strip is $2\Delta y$. The depth of water is $\frac{11}{6}$ ft, so the total force on an end is

$$\int_0^{11/6}(62.4y)(2\,dy) \approx 209.73 \text{ lb}.$$

(b) On the sides, which are twice as long as the ends, the initial total force is doubled to ≈419.47 lb. When the tank is upended, the depth is doubled to $\frac{11}{3}$ ft, and the force on a side becomes

$\int_0^{11/3}(62.4y)(2)dy \approx 838.93$ lb, which means that the fluid force doubles.

26. 3.75 in. $=\frac{5}{16}$ ft, and 7.75 in. $=\frac{31}{48}$ ft.

Force on a side

$$=\int p\,dA$$
$$=\int_0^{31/48}(64.5y)\left(\frac{5}{16}dy\right) \approx 4.2 \text{ lb}.$$

27. $f(t) = \begin{cases} \frac{1}{12} & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$

$$\int_1^5 \left(\frac{1}{12}\right) dt = \left(\frac{t}{12}\right)\Bigg|_1^5 = \frac{5}{12} - \frac{1}{12} = \frac{1}{3}$$

28. $f(t) = \begin{cases} \frac{1}{12} & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$

$$\int_3^6 \left(\frac{1}{12}\right) dt = \left(\frac{t}{12}\right)\Bigg|_3^6 = \frac{6}{12} - \frac{3}{12} = \frac{1}{4}$$

29. Since $\sigma = 2$, then 68% of the calls are answered in 0–4 minutes.

30. Use $f(x) = \dfrac{1}{100\sqrt{2\pi}} e^{-(x-498)^2/(2\cdot 100^2)}$

(a) $\int_{400}^{500} f(x)\,dx \approx 0.34(34\%)$

(b) Take 1000 as a conveniently high upper limit: $\int_{700}^{1000} f(x)dx \approx 0.0217$, which means about 0.0217 (300) ≈ 6.5 people

31. **(a)** 0.5 (50%), since half of a normal distribution lies below the mean.

(b) Use NINT to find $\int_{63}^{65} f(x)\,dx$, where

$$f(x) = \frac{1}{3.2\sqrt{2\pi}} e^{-(x-63.4)^2/(2\cdot 3.2^2)}.$$

The result is ≈0.24 (24%).

(c) 6 ft $= 72$ in. Pick 82 in. as a conveniently high upper limit and with NINT, find $\int_{72}^{82} f(x)\,dx$. The result is ≈0.0036 (0.36%).

(d) 0 if we assume a continuous distribution.

32. Integration is a good approximation to the area (which represents the probability), since the area is a kind of Riemann sum.

33. The proportion of lightbulbs that last between 100 and 800 hours.

34. False; three times as much work is required.

35. True; the force against each vertical side is 842.4 lbs.

36. E; $\int_0^5 (350x)\,dx = 175x^2\Big|_0^5 = 4375$ J.

37. D; $W = \int_0^{20} (50 - 2.5y)dy$

$$= [50y - 1.25y^2]_0^{20}$$
$$= 500\,\text{J}$$

38. B; $F = 200 = kx \Rightarrow k = 4000.$

$$W = \int_0^{0.05} 4000x\,dx = [2000x^2]_0^{0.05} = 5\text{ J}$$

39. E; $\int_0^{12} 62.4(8^2\pi)(12 - y)dy = 903{,}331$ ft-lb.

40. $\int_{6,370,000}^{35,780,000} \dfrac{1000MG}{r^2}dr$

$$= 1000MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000},$$

which for $M = 5.975 \times 10^{24}$, $G = 6.6726 \times 10^{-11}$ evaluates to $\approx 5.1446 \times 10^{10}$ J.

41. **(a)** The distance goes from 2 m to 1 m. The work by an external force equals the work done by repulsion in moving the electrons from a 1-m distance to a 2-m distance:

$$\text{Work} = \int_1^2 \frac{23 \times 10^{-29}}{r^2}dr$$
$$= 23 \times 10^{-29}\left[-\frac{1}{r}\right]_1^2$$
$$= 1.15 \times 10^{-28}\text{ J}$$

(b) Again, find the work done by the fixed electrons in pushing the third one way. The total work is the sum of the work by each fixed electron. The changes in distance are 4 m to 6 m and 2 m to 4 m, respectively.
Work

$$= \int_4^6 \frac{23 \times 10^{-29}}{r^2}dr + \int_2^4 \frac{23 \times 10^{-29}}{r^2}dr$$
$$= 23 \times 10^{-29}\left(\left[-\frac{1}{r}\right]_4^6 + \left[-\frac{1}{r}\right]_2^4\right)$$
$$\approx 7.6667 \times 10^{-29}\text{ J}.$$

42. $F = m\left(\frac{dv}{dt}\right) = mv\left(\frac{dv}{dx}\right)$, so

$$W = \int_{x_2}^{x_2} F(x)\,dx$$
$$= \int_{x_1}^{x_2} mv\left(\frac{dv}{dx}\right)dx$$
$$= \int_{v_1}^{v_2} mv\,dv$$
$$= \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_1^{\,2}$$

43. Since initial velocity = 0,

Work = Change in kinetic energy $= \frac{1}{2}mv_2^2$.

$$m = \frac{2\text{ oz}}{32\text{ ft/sec}^2} = \frac{\frac{1}{8}\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{256}\text{ slug, so}$$

$$\text{Work} = \frac{1}{2}\left(\frac{1}{256}\right)(160)^2 = 50\text{ ft-lb.}$$

44. $0.3125\text{ lb} = \frac{0.3125\text{ lb}}{32\text{ ft/sec}^2} = 0.009765625$ slug,

and

$$90\text{ mph} = 90\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ hr}}{3600\text{ sec}}\right)$$
$$= 132\text{ ft/sec, so}$$

Work = change in kinetic energy

$$= \frac{1}{2}(0.009765625)(132)^2$$
$$\approx 85.1\text{ ft-lb.}$$

45. $1.6\text{ oz} = 1.6\text{ oz}\frac{\left(\frac{1\text{ lb}}{16\text{ oz}}\right)}{(32\text{ ft/sec}^2)} = 0.003125$ slug,

so Work $= \frac{1}{2}(0.003125)(280)^2 = 122.5$ ft-lb.

46. $2\text{ oz} = 2\text{ oz}\frac{\left(\frac{1\text{ lb}}{16\text{ oz}}\right)}{(32\text{ ft/sec}^2)} = \frac{1}{256}$ slug, and

$$124\text{ mph} = 124\text{ mph}\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ hr}}{3600\text{ sec}}\right)$$
$$= 181.867\text{ ft/sec,}$$

so Work $= \frac{1}{2}\left(\frac{1}{256}\right)(181.867)^2 \approx 64.6$ ft-lb.

47. $14.5\text{ oz} = 14.5\text{ oz}\frac{\left(\frac{1\text{ lb}}{16\text{ oz}}\right)}{(32\text{ ft/sec}^2)} \approx 0.02832$ slug,

so Work $\approx \frac{1}{2}(0.02832)(88)^2 \approx 109.7$ ft-lb.

48. $6.5\text{ oz} = 6.5\text{ oz}\frac{\left(\frac{1\text{ lb}}{16\text{ oz}}\right)}{(32\text{ ft/sec}^2)} \approx 0.01270$ slug, so

Work $\approx \frac{1}{2}(0.01270)(132)^2 \approx 110.6$ ft-lb.

49. $2\text{ oz} = \frac{1}{8}\text{ lb} = \frac{1}{256}$ slug. Compression energy

of spring $= \frac{1}{2}ks^2 = \frac{1}{2}(18)\left(\frac{1}{4}\right)^2 = 0.5625$ ft-lb,

and final height is given by

$mgh = 0.5625$ ft-lb, so $h = \frac{0.5625}{\left(\frac{1}{256}\right)(32)} = 4.5$ ft.

Quick Quiz Sections 8.4 and 8.5

1. A; $y = \int\sqrt{16x^6}\,dx$

$$y = \int 4x^3dx$$
$$y = x^4$$
$$y = (x^4 - 1) + 4 = x^4 + 3$$

2. D; $x = \frac{1}{4}t^4$

$$\frac{dx}{dt} = t^3$$
$$y = t^3$$
$$\frac{dy}{dt} = 3t^2$$
$$\int_0^2\left((t^3)^2 + (3t^2)^2\right)^{1/2}dt = \int_0^2\sqrt{t^6 + 9t^4}\,dt$$

3. C; $\int_{-2}^{2}\left(2\sqrt{4 - x^2}\right)^2 dx = \int_{-2}^{2}(16 - 4x^2)dx$

$$= \left(16x - \frac{4}{3}x^3\right)_{-2}^{2}$$
$$= \frac{128}{3}\text{ in.}^3$$

4. **(a)** $\sum_{i=1}^{n} 62.4h \cdot 2\Delta h$

(b) $\int_0^{1.5} 62.4h \cdot 2dh = 62.4h^2\Big|_0^{1.5} = 140.4$ 1bs.

(c) $\int_0^1 62.4h \cdot 2\sqrt{1-h^2}\,dh = 41.6$ 1bs

Chapter 8 Review Exercises (pp. 430–433)

1. $\int_0^5 v(t)\,dt = \int_0^5 (t^2 - 0.2t^3)\,dt$

$= \left[\frac{1}{3}t^3 - 0.05t^4\right]_0^5 \approx 10.417\,\text{ft}$

2. $\int_0^7 c(t)\,dt = \int_0^7 (4 + 0.001t^4)\,dt$

$= [4t + 0.0002t^5]_0^7 \approx 31.361\,\text{gal}$

3. $\int_0^{100} B(x)\,dx = \int_0^{100} (21 - e^{0.03x})\,dx$

$\approx [21x - 33.333e^{0.03x}]_0^{100}$

≈ 1464

4. $\int_0^2 p(x)\,dx = \int_0^2 (11 - 4x)\,dx$

$= [11x - 2x^2]_0^2$

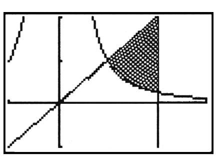
$= 14\,g$

5. $\int_0^{24} E(t)\,dt = \int_0^{24} 300\left(2 - \cos\left(\frac{\pi t}{12}\right)\right)dt$

$= 300\left[2t - \frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right)\right]_0^{24}$

$= 14,400$

6.



[–1, 3] by [–1, 2]

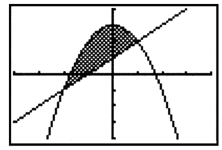
The curves intersect at $x = 1$. The area is

$\int_1^2 \left[x - \frac{1}{x^2}\right]dx = \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2$

$= \left[\left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right)\right]$

$= 1.$

7.



[–4, 4] by [–4, 4]

The curves intersect at $x = -2$ and $x = 1$. There area is

$\int_{-2}^1 [3 - x^2 - (x+1)]\,dx$

$= \int_{-2}^1 (-x^2 - x + 2)\,dx$

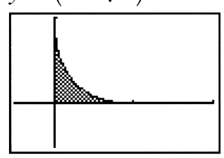
$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^1$

$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$

$= \frac{9}{2}.$

8. $\sqrt{x} + \sqrt{y} = 1$ implies

$y = \left(1 - \sqrt{x}\right)^2 = 1 - 2\sqrt{x} + x.$



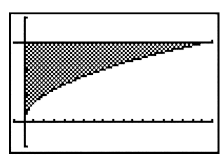
[–0.5, 2] by [–0.5, 1]

The area is

$\int_0^1 \left(1 - 2\sqrt{x} + x\right)dx = \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1$

$= \frac{1}{6}.$

9. $x = 2y^2$ implies $y = \sqrt{\frac{x}{2}}.$



[–1, 19] by [–1, 4]

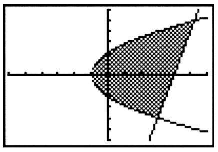
The curves intersect at $x = 18$. The area is

$\int_0^{18} \left(3 - \sqrt{\frac{x}{2}}\right)dx = \left[3x - \frac{4}{3}\left(\frac{x}{2}\right)^{3/2}\right]_0^{18} = 18,$ or

$\int_0^3 2y^2\,dy = \left[\frac{2}{3}y^3\right]_0^3 = 18.$

10. $4x = y^2 - 4$ implies $x = \frac{1}{4}y^2 - 1$, and

$4x = y + 16$ implies $x = \frac{1}{4}y + 4$.

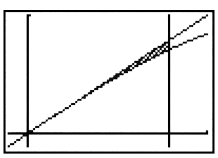


[–6, 6] by [–6, 6]

The curves intersect at (3, –4) and (5.25, 5). The area is

$$\int_{-4}^{5}\left[\left(\frac{1}{4}y+4\right)-\left(\frac{1}{4}y^2-1\right)\right]dy$$
$$=\int_{-4}^{5}\left(-\frac{1}{4}y^2+\frac{1}{4}y+5\right)dy$$
$$=\left[-\frac{1}{12}y^3+\frac{1}{8}y^2+5y\right]_{-4}^{5}$$
$$=\frac{425}{24}-\left(-\frac{38}{3}\right)$$
$$=\frac{243}{8}$$
$$=30.375.$$

11.

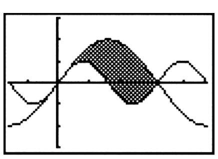


[–0.1, 1] by [–0.1, 1]

The area is

$$\int_0^{\pi/4}(x-\sin x)\,dx=\left[\frac{1}{2}x^2+\cos x\right]_0^{\pi/4}$$
$$=\frac{\pi^2}{32}+\frac{\sqrt{2}}{2}-1$$
$$\approx 0.0155.$$

12.

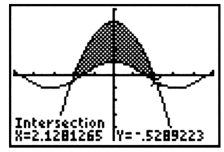


$\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ by $[-3, 3]$

The area is $\int_0^{\pi}(2\sin x-\sin 2x)\,dx$

$$=\left[-2\cos x+\frac{1}{2}\cos 2x\right]_0^{\pi}$$
$$=4.$$

13.

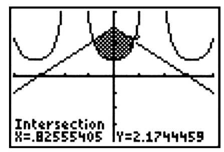


[–5, 5] by [–5, 5]

The curves intersect at $x \approx \pm 2.1281$. The area is $\int_{-2.1281}^{2.1281}(4-x^2-\cos x)\,dx$

$$=\left[4x-\frac{1}{3}x^3-\sin x\right]_{-2.1281}^{2.1281}$$
$$\approx 8.9023.$$

14.



[–4, 4] by [–4, 4]

The curves intersect at $x \approx \pm 0.8256$. The area is $\int_{-0.8256}^{0.8256}(3-|x|-\sec^2 x)\,dx$

$$=2\int_0^{0.8256}(3-x-\sec^2 x)dx$$
$$=2\left[3x-\frac{1}{2}x^2-\tan x\right]_0^{0.8256}$$
$$\approx 2.1043.$$

15. Solve $1 + \cos x = 2 - \cos x$ for the x-values at the two ends of the region: $x = 2\pi \pm \frac{\pi}{3}$, i.e., $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$. Use the symmetry of the area:

$$2\int_{2\pi}^{7\pi/3}[(1+\cos x)-(2-\cos x)]\,dx$$
$$=2\int_{2\pi}^{7\pi/3}(2\cos x-1)\,dx$$
$$=2[2\sin x-x]_{2\pi}^{7\pi/3}$$
$$=2\sqrt{3}-\frac{2}{3}\pi\approx 1.370.$$

16. $\int_{\pi/3}^{5\pi/3}[(2-\cos x)-(1+\cos x)]\,dx$

$$=\int_{\pi/3}^{5\pi/3}(1-2\cos x)\,dx$$
$$=[1-2\sin x]_{\pi/3}^{5\pi/3}$$
$$=2\sqrt{3}+\frac{4}{3}\pi\approx 7.653$$

17. Solve $x^3 - x = \dfrac{x}{x^2+1}$ to find the intersection points at $x=0$ and $x=\pm 2^{1/4}$. Then use the area's symmetry: the area is

$$2\int_0^{2^{1/4}} \left[\frac{x}{x^2+1} - (x^3 - x)\right] dx$$
$$= 2\left[\frac{1}{2}\ln(x^2+1) - \frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^{2^{1/4}}$$
$$= \ln\left(\sqrt{2}+1\right) + \sqrt{2} - 1 \approx 1.2956.$$

18. Use the intersect function on a graphing calculator to determine that the curves intersect at $x \approx 1.8933$.

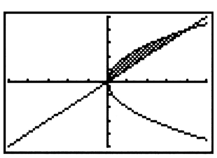
The area is $\displaystyle\int_{-1.8933}^{1.8933} \left(3^{1-x^2} - \frac{x^2-3}{10}\right) dx,$

which using NINT evaluates to ≈ 5.7312.

19. Use the x- and y-axis symmetries of the area:
$4\int_0^{\pi} x\sin x\, dx = 4[\sin x - x\cos x]_0^{\pi} = 4\pi.$

20. A cross section has radius $r = 3x^4$ and area $A(x) = \pi r^2 = 9\pi x^8$.
$v = \int_{-1}^{1} 9\pi x^8\, dx = \pi[x^9]_{-1}^{1} = 2\pi$

21.



[–5, 5] by [–5, 5]

The graphs intersect at (0, 0) and (4, 4).

(a) Use cylindrical shells. A shell has radius y and height $y - \dfrac{y^2}{4}$. The total volume is

$$\int_0^4 2\pi(y)\left(y - \frac{y^2}{4}\right) dy$$
$$= 2\pi\int_0^4 \left(y^2 - \frac{y^3}{4}\right) dy$$
$$= 2\pi\left[\frac{1}{3}y^3 - \frac{1}{16}y^4\right]_0^4$$
$$= \frac{32\pi}{3}.$$

(b) Use cylindrical shells. A shell has radius x and height $2\sqrt{x} - x$. The total volume is

$$\int_0^4 2\pi(x)\left(2\sqrt{x} - x\right) dx$$
$$= 2\pi\int_0^4 (2x^{3/2} - x^2)\, dx$$
$$= 2\pi\left[\frac{4}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^4$$
$$= \frac{128\pi}{15}.$$

(c) Use cylindrical shells. A shell has radius $4 - x$ and height $2\sqrt{x} - x$. The total volume is

$$\int_0^4 2\pi(4-x)\left(2\sqrt{x} - x\right) dx$$
$$= 2\pi\int_0^4 \left(8\sqrt{x} - 4x - 2x^{3/2} + x^2\right) dx$$
$$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{1}{3}x^3\right]_0^4$$
$$= \frac{64\pi}{5}.$$

(d) Use cylindrical shells. A shell has radius $4 - y$ and height $y - \dfrac{y^2}{4}$. The total volume is

$$\int_0^4 2\pi(4-y)\left(y - \frac{y^2}{4}\right) dy$$
$$= 2\pi\int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{1}{16}y^4\right]_0^4$$
$$= \frac{32\pi}{3}.$$

22. (a) Use disks. The volume is

$$\pi\int_0^2 \left(\sqrt{2y}\right)^2 dy = \pi\int_0^2 2y\, dy = \pi y^2\Big|_0^2 = 4\pi.$$

(b) $\pi\int_0^k 2y\, dy = \pi y^2\Big|_0^k = \pi k^2$

(c) Since $V = \pi k^2$, $\dfrac{dV}{dt} = 2\pi k\dfrac{dk}{dt}$.

When $k = 1$,

$\dfrac{dk}{dt} = \dfrac{1}{2\pi k}\dfrac{dV}{dt} = \left(\dfrac{1}{2\pi}\right)(2) = \dfrac{1}{\pi}$, so the

depth is increasing at the rate of $\frac{1}{\pi}$ unit per second.

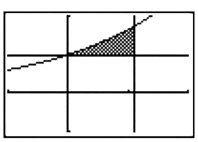
23. The football is a solid of revolution about the x-axis. A cross section has radius $\sqrt{12\left(1-\frac{4x^2}{121}\right)}$ and area $\pi r^2 = 12\pi\left(1-\frac{4x^2}{121}\right)$. The volume is, given the symmetry,

$$2\int_0^{11/2} 12\pi\left(1-\frac{4x^2}{121}\right)dx$$
$$= 24\pi\int_0^{11/2}\left(1-\frac{4x^2}{121}\right)dx$$
$$= 24\pi\left[x-\left(\frac{2}{11}\right)^2\left(\frac{1}{3}\right)x^3\right]_0^{11/2}$$
$$= 24\pi\left[\frac{11}{2}-\frac{11}{6}\right]$$
$$= 88\pi \approx 276\,\text{in}^3.$$

24. The width of a cross section is 2 sin x, and the area is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\sin^2 x$. The volume is

$$\int_0^{\pi}\frac{1}{2}\pi\sin^2 x\,dx = \frac{\pi}{2}\left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{4}.$$

25.



[–1, 2] by [–1, 2]

Use washer cross sections. A washer has inner radius $r = 1$, outer radius $R = e^{x/2}$, and area $\pi(R^2 - r^2) = \pi(e^x - 1)$.

The volume is

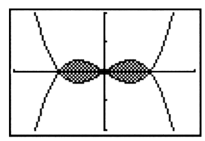
$$\int_0^{\ln 3}\pi(e^x-1)\,dx = \pi\left[e^x - x\right]_0^{\ln 3}$$
$$= \pi(3-\ln 3-1)$$
$$= \pi(2-\ln 3).$$

26. Use cylindrical shells. Taking the hole to be vertical, a shell has radius x and height $2\sqrt{2^2-x^2}$. The volume of the piece cut out is

$$\int_0^{\sqrt{3}} 2\pi(x)\left(2\sqrt{2^2-x^2}\right)dx$$
$$= 2\pi\int_0^{\sqrt{3}} 2x\sqrt{4-x^2}\,dx$$
$$= 2\pi\left[-\frac{2}{3}(4-x^2)^{3/2}\right]_0^{\sqrt{3}}$$
$$= -\frac{4}{3}\pi(1-8)$$
$$= \frac{28\pi}{3} \approx 29.3215\,\text{ft}^3.$$

27. The curve crosses the x-axis at ± 3. $y' = -2x$, so the length is $\int_{-3}^{3}\sqrt{1+(-2x)^2}\,dx = \int_{-3}^{3}\sqrt{1+4x^2}\,dx$, which using NINT evaluates to ≈ 19.4942.

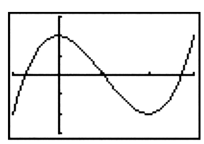
28.



[–2, 2] by [–2, 2]

The curves intersect at $x = 0$ and $x = \pm 1$. Use the graphs' x- and y-axis symmetry: $\frac{d}{dx}(x^3 - x) = 3x^2 - 1$, and the total perimeter is $4\int_0^1\sqrt{1+(3x^2-1)^2}\,dx$, which using NINT evaluates to ≈ 5.2454.

29.



[–1, 3] by [–3, 3]

$y' = 3x^2 - 6x$ equals zero when $x = 0$ or 2.The maximum is at $x = 0$, the minimum at $x = 2$. The distance between them along the curve is $\int_0^2\sqrt{1+(3x^2-6x)^2}\,dx$, which using NINT evaluates to ≈ 4.5920. The time taken is about $\frac{4.5920}{2} = 2.296\,\text{sec}$.

30. If (b) were true, then the curve $y = k\sin x$ would have to get vanishingly short as k approaches zero. Since in fact the curve's length approaches 2π instead, (b) is false and (a) is true.

31. $F'(x) = \sqrt{x^4 - 1}$, so

$$\int_2^5 \sqrt{1+(F'(x))^2}\,dx = \int_2^5 \sqrt{x^4}\,dx = \int_2^5 x^2\,dx = \left[\frac{1}{3}x^3\right]_2^5 = 39.$$

32. **(a)** (100 N)(40 m) = 4000 J

(b) When the end has traveled a distance y, the weight of the remaining portion is $(40 - y)(0.8) = 32 - 0.8y$.
The total work to lift the rope is

$$\int_0^{40} (32 - 0.8y)\,dy = [32y - 0.4y^2]_0^{40} = 640\text{ J}.$$

(c) 4000 + 640 = 4640 J

33. The weight of the water at elevation x (starting from $x = 0$) is

$$(800)(8)\left(\frac{4750 - \frac{x}{2}}{4750}\right) = \frac{128}{95}\left(4750 - \frac{1}{2}x\right).$$

The total work is

$$\int_0^{4750} \frac{128}{95}\left(4750 - \frac{1}{2}x\right)dx = \frac{128}{95}\left[4750x - \frac{1}{4}x^2\right]_0^{4750} = 22{,}800{,}000 \text{ ft-lb}.$$

34. $F = ks$, so $k = \frac{F}{s} = \frac{80}{0.3} = \frac{800}{3}$ N/m. Then

$$\text{Work} = \int_0^{0.3} \frac{800}{3}x\,dx = \left[\frac{800}{6}x^2\right]_0^{0.3} = 12 \text{ J}.$$

To stretch the additional meter,

$$\text{Work} = \int_{0.3}^{1.3} \frac{800}{3}x\,dx = \left[\frac{800}{6}x^2\right]_{0.3}^{1.3} \approx 213.3 \text{ J}.$$

35. The same amount of work is done, but gravity supplies the downhill force, so less work is done by the person.

36. The radius of a horizontal cross section is $\sqrt{8^2 - y^2}$, where y is distance below the rim. The area is $\pi(64 - y^2)$, the weight is $0.04\pi(64 - y^2)\Delta y$, and the work to lift it over the rim is $0.04\pi(64 - y^2)(y)\Delta y$. The total work is

$$\int_2^8 0.04\pi y(64 - y^2)\,dy = 0.04\pi\int_2^8 (64y - y^3)\,dy = 0.04\pi\left[32y^2 - \frac{1}{4}y^4\right]_2^8 = 36\pi \approx 113.097 \text{ in-lb}.$$

37. The width of a thin horizontal strip is $2(2y) = 4y$, and the force against it is $80(2 - y)4y\,\Delta y$. The total force is

$$\int_0^2 320y(2 - y)\,dy = 320\int_0^2 (-y^2 + 2y)\,dy = 320\left[-\frac{1}{3}y^3 + y^2\right]_0^2 = \frac{1280}{3} \approx 426.67 \text{ lb}.$$

38. $5.75 \text{ in.} = \frac{23}{48}$ ft, $3.5 \text{ in.} = \frac{7}{24}$ ft, and $10 \text{ in.} = \frac{5}{6}$ ft.

For the base,

$$\text{Force} = 57\left(\frac{23}{48}\times\frac{7}{24}\times\frac{5}{6}\right) \approx 6.6385 \text{ lb}.$$

For the front and back,

$$\text{Force} = \int_0^{5/6} 57\left(\frac{7}{24}\right)y\,dy = \frac{399}{24}\left[\frac{1}{2}y^2\right]_0^{5/6} \approx 5.7726 \text{ lb}.$$

For the sides,

$$\text{Force} = \int_0^{5/6} 57\left(\frac{23}{48}\right)y\,dy = \frac{1311}{48}\left[\frac{1}{2}y^2\right]_0^{5/6} \approx 9.4835 \text{ lb}.$$

39. A square's height is $y = \left(\sqrt{6} - \sqrt{x}\right)^2$, and its area is $y^2 = \left(\sqrt{6} - \sqrt{x}\right)^4$. The volume is

$$\int_0^6 (\sqrt{6} - \sqrt{x})^4\,dx = \int_0^6 \left(36 - 24\sqrt{6}x^{1/2} + 36x - 4\sqrt{6}x^{3/2} + x^2\right)dx = \left[36x - 16\sqrt{6}x^{3/2} + 18x^2 - 1.6\sqrt{6}x^{5/2} + \frac{1}{3}x^3\right]_0^6 = 14.4$$

40. Choose 50 cm as a conveniently large upper limit.

$\int_{20}^{50} \frac{1}{3.4\sqrt{2\pi}} e^{-(x-17.2)^2/(2\cdot 3.4^2)}\,dx$ evaluates, using NINT, to $\approx 0.2051(20.5\%)$.

41. Answers will vary. Find μ, then use the fact that 68% of the class is within σ of μ to find σ, and then choose a conveniently large number b and calculate $\int_{10}^{b} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx$.

42. Use $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(a) $\int_{-1}^{1} f(x)\,dx$ evaluates, using NINT, to $\approx 0.6827(68.27\%)$.

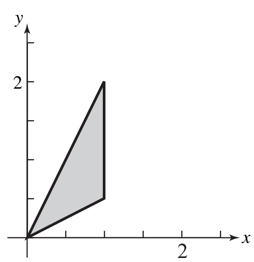
(b) $\int_{-2}^{2} f(x)\,dx \approx 0.9545(95.45\%)$

$\int_{-3}^{3} f(x)\,dx \approx 0.9973(99.73\%)$

43. Because $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

The total area gives the probability that the variable takes on one of its possible values. Since the variable must take on some value, the probability must be 1.

44.

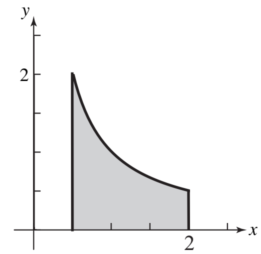


A shell has radius x and height $2x - \frac{x}{2} = \frac{3}{2}x$.

The total volume is

$\int_0^1 2\pi(x)\left(\frac{3}{2}x\right)dx = \pi[x^3]_0^1 = \pi$.

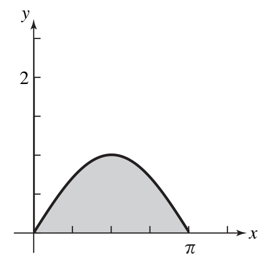
45.



A shell has radius x and height $\frac{1}{x}$. The total volume is

$\int_{1/2}^{2} 2\pi(x)\left(\frac{1}{x}\right)dx = \int_{1/2}^{2} 2\pi\,dx = [2\pi x]_{1/2}^{2} = 3\pi$.

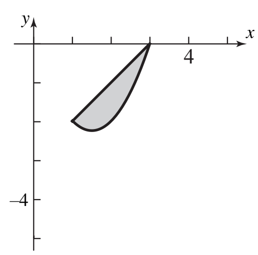
46.



A shell has radius x and height $\sin x$. The total volume is

$$\int_0^{\pi} 2\pi(x)(\sin x)\,dx = 2\pi\left[\sin x - x\cos x\right]_0^{\pi} = 2\pi^2.$$

47.



The curves intersect at $x = 1$ and $x = 3$. A shell has radius x and height $x - 3 - (x^2 - 3x) = -x^2 + 4x - 3$. The total volume is

$$\int_1^3 2\pi(x)(-x^2 + 4x - 3)\,dx$$
$$= 2\pi\int_1^3 (-x^3 + 4x^2 - 3x)\,dx$$
$$= 2\pi\left[-\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3$$
$$= \frac{16\pi}{3}.$$

48. Use the intersect function on a graphing calculator to determine that the curves intersect at $x \approx \pm 1.8933$. A shell has radius x and height $3^{1-x^2} - \dfrac{x^2-3}{10}$. The volume, which is calculated using the *right half* of the area, is $\int_0^{1.8933} 2\pi(x)\left(3^{1-x^2} - \dfrac{x^2-3}{10}\right)dx$, which using NINT evaluates to ≈ 9.7717.

49. **(a)** $y = -\dfrac{5}{4}(x+2)(x-2) = 5 - \dfrac{5}{4}x^2$

(b) Revolve about the line $x = 4$, using cylindrical shells. A shell has radius $4 - x$ and height $5 - \dfrac{5}{4}x^2$. The total volume is

$$\begin{aligned}\int_{-2}^{2} 2\pi(4-x)\left(5 - \frac{5}{4}x^2\right)dx &= 10\pi\int_{-2}^{2}\left(\frac{1}{4}x^3 - x^2 - x + 4\right)dx \\ &= 10\pi\left[\frac{1}{16}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-2}^{2} \\ &= \frac{320}{3}\pi \approx 335.1032 \text{ in}^3.\end{aligned}$$

50. Since $\dfrac{dL}{dx} = \dfrac{1}{x} + f'(x)$ must equal $\sqrt{1+(f'(x))^2}$, $1+(f'(x))^2 = \dfrac{1}{x^2} + \dfrac{2}{x}f'(x) + f'(x)^2$, and $f'(x) = \dfrac{1}{2}x - \dfrac{1}{2x}$. Then $f(x) = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x + C$, and the requirement to pass through (1, 1) means that $C = \dfrac{3}{4}$. The function is $f(x) = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x + \dfrac{3}{4} = \dfrac{x^2 - 2\ln x + 3}{4}$.

51. $y' = \sec^2 x$, so the area is $\int_0^{\pi/4} 2\pi(\tan x)\sqrt{1+(\sec^2 x)^2}\, dx$, which using NINT evaluates to ≈ 3.84.

52. $x = \dfrac{1}{y}$ and $x' = -\dfrac{1}{y^2}$, so the area is $\int_1^2 2\pi\left(\dfrac{1}{y}\right)\sqrt{1+\left(-\dfrac{1}{y^2}\right)^2}\, dy$, which using NINT evaluates to ≈ 5.02.

53. **(a)** The two curves intersect at $x \approx 1.2237831$. Store this value as A.
Area $= \int_0^A (2 + \sin x - \sec x)\, dx = 1.366.$

(b) Volume $= \int_0^A \pi\left((2+\sin x)^2 - (\sec x)^2\right)dx$
$= 16.404.$

(c) Volume $= \int_0^A (2 + \sin x - \sec x)^2\, dx$
$= 1.629.$

54. **(a)** Average temp $= \dfrac{1}{14-6}\int_6^{14}\left(80 - 10\cos\left(\dfrac{\pi t}{12}\right)\right)dt$
$= 87° F.$

(b) $F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right) \geq 78$ for $5.2308694 \leq t \leq 18.769131$.

Store these two values as A and B.

(c) $\text{Cost } = 0.05\int_A^B \left(80 - 10\cos\left(\frac{\pi t}{12}\right) - 78\right) dt$

≈ 5.10

The cost was about \$5.10.

55. (a) $\int_9^{17} \frac{15600}{(t^2 - 24t + 160)} dt \approx 6004$ people.

(b) $15\int_9^{17} \frac{15600}{(t^2 - 24t + 160)} dt + 11\int_{17}^{23} \frac{15600}{(t^2 - 24t + 160)} dt \approx 104{,}048$ dollars

(c) $H'(17) = E(17) - L(17) \approx -380$ people. $H(17)$ is the number of people in the park at 5:00, and $H'(17)$ is the rate at which the number of people in the park is changing at 5:00.

(d) When $H'(t) = E(t) - L(t) = 0$; that is, at $t = 15.795$